LMI-Based Synthesis for Distributed Event-Based State Estimation

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***Abstract*— This paper presents an LMI-based synthesis procedure for distributed event-based state estimation. Multiple agents observe and control a dynamic process by sporadically exchanging data over a broadcast network according to an event-based protocol. In previous work [1], the synthesis of event-based state estimators is based on a centralized design. In that case three different types of communication are required: event-based communication of measurements, periodic reset of all estimates to their joint average, and communication of inputs. The proposed synthesis problem eliminates the communication of inputs as well as the periodic resets (under favorable circumstances) by accounting explicitly for the distributed structure of the control system.**

I. Introduction

Present day control systems are typically implemented on digital hardware with periodic exchange of information between the system's various components (sensors, actuators, controllers). While the design of digital control systems with periodic communication is well understood, it comes with a fundamental limitation: system resources such as computation and communication are used at predetermined time instants irrespective of the current state of the system, or the information content of the data to be passed between the components. Because of this, aperiodic or event-based communication has recently gained popularity as an alternative to periodic communication for control, estimation, and optimization (see overview articles [2]–[5]). While the seminal papers [6], [7] and following early work mostly concerned fundamental questions of event-based communication for a single feedback loop, the main potential of event-based strategies arguably lies with *distributed* problems, where multiple components share common resources such as a communication network.

We consider event-based communication for a distributed estimation and control problem, where multiple sensor-actuator-agents observe and control a dynamic system and communicate with each other via a common bus (see Fig. 1). In previous work [1], [8], a method for distributed event-based state estimation was proposed for such a system. The event-based estimator implemented on each agent has two main components: the event generator (EG), which triggers the sporadic communication of local sensor measurements to all other agents, and the state estimator, which fuses the measurement data received from the network to compute an estimate of the state. Predictions from the estimator are used in the EG for making the transmit decisions, and its state estimates can be used for feedback control (cf. Fig. 1).

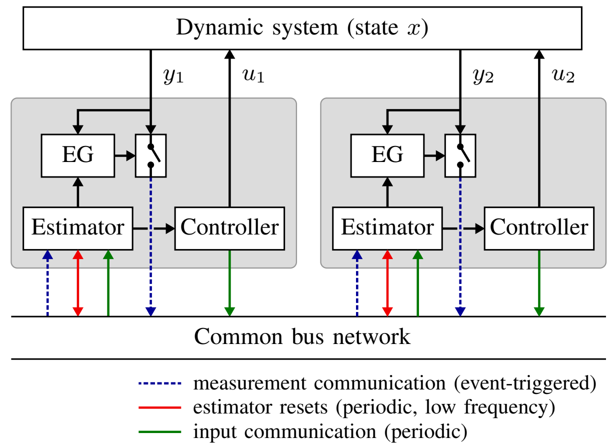

Fig. 1. Networked control system considered in this paper (shown here with two agents as an example). Each agent observes part of the system state x through local sensors y_i. The *Estimator* on each agent computes an estimate of x, the event generator (*EG*) triggers the communication of local sensor measurements based on a threshold logic, and the *Controller* computes commands u_i for the local actuator. Event-triggered communication is indicated by dashed arrows, while periodic communication is shown by solid ones. In [1], three types of communication are required: event-based communication of measurements (blue), periodic resetting of all estimates to their average (red), and periodic communication of inputs (green). In this paper, the structure of the event-based estimation is kept, but communication of inputs and (under favorable conditions) estimator resets are eliminated.

The methods [1], [8] are effective in reducing measurement communication: sensor data is exchanged between the agents only when necessary to meet a certain estimation performance. However, they also require two additional types of inter-agent communication (cf. Fig. 1): (i) periodic exchange of control inputs and (ii) periodic (albeit infrequent) exchange and reset of estimates, to guarantee stability in case of differences between the agents' estimates, for example, from imperfect communication or inaccurate initialization.

In this paper, a modified design is proposed that alleviates (and ideally avoids) these additional communication requirements. We pose the design of the estimator gains as a linear matrix inequality (LMI) optimization problem, which under favorable conditions guarantees stability of the event-based estimation in the presence of differences between the agents, without periodic resetting of the estimates. For problems, where the LMI-design cannot guarantee stability without resetting, it lends itself to an event-driven resetting mechanism as an alternative to the periodic resets in [1].

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Moreover, the periodic exchange of control inputs is avoided by each agent locally estimating the full input vector from its state estimate. Ideally, only event-triggered communication of measurements remains with the modified scheme.

Outline of the paper: After a brief review of some related work, the distributed event-based estimation method from [1] is summarized in Sec. II. LMI-based stability conditions for the event-based estimation problem are derived in Sec. III, followed by the corresponding synthesis problem for the estimator gains in Sec. IV. The improved event-based estimator design is illustrated with a simulation example in Sec. V, and the paper concludes with remarks in Sec. VI.

Related work: LMI-based designs for distributed event-based estimation are discussed in [9], [10] for different problems, where communication between agents is according to a graph topology. In [9], an LMI-design is proposed for the estimator gains first assuming periodic communication, which is then implemented as an event-based scheme using event-triggers on local state estimates. The authors in [10] consider event-triggers on measurements and formulate an LMI synthesis problem for both estimator gains and triggering levels. Both references do not explicitly consider disturbances on the estimates, and they exclusively treat the estimation problem, while we simultaneously address stability of the distributed event-based control system resulting when local estimates are used for feedback control.

For further references on event-based estimation and control, see surveys [2]–[5], and references therein and in [1].

II. PROBLEM FORMULATION

In this section, the networked dynamic system is introduced (Sec. II-A) along with the distributed event-based estimation and control architecture (Sec. II-B and II-C), and the objective of this paper is formulated (Sec. II-D).

A. Networked System

Consider the discrete-time linear system

$$x(k) = Ax(k-1) + Bu(k-1) + v(k-1) \tag{1}$$

$$y(k) = Cx(k) + w(k) \tag{2}$$

with time index k, state $x(k) \in \mathbb{R}^n$, control input $u(k) \in \mathbb{R}^q$, measurement $y(k) \in \mathbb{R}^p$, disturbances $v(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^p$, and all matrices of corresponding dimensions. The disturbances w and v are assumed bounded; (A, B) and (A, C) are assumed stabilizable and detectable, respectively.

The inputs $u(k)$ and measurements $y(k)$ are decomposed corresponding to N sensor-actuator-agents:

$$B\,u(k-1) = \begin{bmatrix} B_1 & B_2 & \dots & B_N \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_N(k) \end{bmatrix} \tag{3}$$

$$y(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_N(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} x(k) + \begin{bmatrix} w_1(k) \\ w_2(k) \\ \vdots \\ w_N(k) \end{bmatrix} \tag{4}$$

where $u_i(k) \in \mathbb{R}^{q_i}$ is agent i's input and $y_i(k) \in \mathbb{R}^{p_i}$ its measurement. Note that agents may be heterogeneous and, in particular, their input and output dimensions q_i and p_i may differ (including the case $q_i = 0$ or $p_i = 0$, i.e. no actuator or sensor is present for agent i). Local stabilizability or detectability is *not* required; that is, (A, B_i) may be not stabilizable, and (A, C_i) may be not detectable.

The agents can exchange sensor data $y_i(k)$ with each other over a broadcast network; that is, if one agent communicates, all other agents will receive the data. The event-based mechanism determining when sensor data is exchanged will be made precise in the next subsection. The agents do not share input data $u_i(k)$ with each other, which is in contrast to [1], [8]. Agents are assumed to be synchronized in time, and network communication is assumed to be instantaneous and without delay.

We assume that a state-feedback controller

$$u(k) = Fx(k) \tag{5}$$

is given, which renders $A+BF$ asymptotically stable (magnitude of all eigenvalues strictly less than one). The controller can be designed using standard methods, see e.g. [11].

B. Distributed Event-Based State Estimation

The distributed, event-based state estimation architecture from [1], [8] is briefly summarized next. It serves as the starting point for the LMI-based design of estimator gains proposed later. According to this architecture, each agent implements an *event generator*, which makes the transmit decision for the local measurement, and a *state estimator*, which computes a local state estimate.

1) Event Generator: The event generator on agent i decides whether or not the local measurement $y_i(k)$ is broadcast to all other agents by applying the decision rule:

$$\text{transmit } y_i(k) \quad \Leftrightarrow \quad ||y_i(k) - C_i\hat{x}_i(k|k-1)||_2 \geq \delta_i \tag{6}$$

where $\delta_i \geq 0$ is a design parameter, $\hat{x}_i(k|k-1)$ is agent i's prediction of the state $x(k)$ based on measurements until time $k-1$ (to be made precise below), and $C_i\hat{x}_i(k|k-1)$ is agent i's prediction of its measurement $y_i(k)$. Hence, measurement $y_i(k)$ is transmitted if, and only if, its prediction from the previous estimate deviates by more than the tolerable threshold δ_i. Let

$$I(k) := \{i \mid 1 \leq i \leq N,\ ||y_i(k) - C_i\hat{x}_i(k|k-1)||_2 \geq \delta_i\} \tag{7}$$

denote the index set of measurements transmitted at time k.

2) State Estimator: Let $\hat{x}_i(k) = \hat{x}_i(k|k)$ denote agent i's estimate of the system state $x(k)$ computed from all measurements $I(\ell)$ for $\ell = 1, \dots, k$. The recursive estimator update of agent i is given by

$$\hat{x}_i(k|k-1) = A\hat{x}_i(k-1|k-1) + B\hat{u}^i(k-1) \tag{8}$$

$$\hat{x}_i(k|k) = \hat{x}_i(k|k-1) + \sum_{j \in I(k)} L_j\big(y_j(k) - C_j\hat{x}_i(k|k-1)\big) + d_i(k) \tag{9}$$

where $\hat{u}^i(k-1) \in \mathbb{R}^q$ is agent i's belief of the input vector $u(k)$, L_j are observer gains to be designed, and $d_i(k)$ represents a disturbance, which is assumed to be bounded. The disturbance d_i has been introduced in [1] to model mismatches between the estimates of the individual agents, which may stem from, for example, unequal initialization, different computation accuracy, or imperfect communication.

Note that the event-based estimator computes the correction (9) from all measurements $I(k)$ that satisfy (6). Hence, the local measurement $y_i(k)$ is used in the update only if it satisfies (6), even though it could actually be used in the update at every step without requiring any additional communication. This scheme was suggested in [8] to ensure that (in the absence of disturbances d_i) all agents have consistent estimates, since all estimates are updated with the same set of measurements $I(k)$. We follow the same basic idea herein. See Sec. VI for a brief discussion on an alternative estimator update law employing local measurements at every step.

C. Distributed Control

The control $u_i(k)$ on agent i is computed according to the distributed law

$$u_i(k) = F_i \hat{x}_i(k) \tag{10}$$

where $F^{\mathrm{T}} = [F_1^{\mathrm{T}}\, F_2^{\mathrm{T}} \ldots F_N^{\mathrm{T}}]$ is the decomposition of the state-feedback gain F in (5) according to the dimensions of $u_i(k)$. In [1], [8], it is assumed that each agent broadcasts its control component $u_i(k)$ at every step, so that each agent knows the full input vector $u(k)$ and can implement (8) with

$$\hat{u}^i(k-1) = u(k-1). \tag{11}$$

Herein, we do *not* require the communication of the inputs $u_i(k)$ and instead use a local estimate of the input vector

$$\hat{u}^i(k-1) = F\hat{x}_i(k-1) \tag{12}$$

in (8). Avoiding the communication of inputs in this way was also suggested and experimentally tested in [12], but stability of this scheme was not formally proven.

D. Objective

In brief, we seek to design observer gains L_j such that the closed-loop system, given by (1), (2), (8), (9), (10), and (12), is stable for bounded disturbances v, w, and d_i.

In [8], closed-loop stability is shown for $d_i \equiv 0$ and all inputs $u_i(k)$ being communicated such that (11) is implemented instead of (12). The estimator gains L_j are obtained in [8] from a straightforward centralized observer design (a Luenberger observer). The only requirement for the design is that the gains stabilize the centralized, full communication case.

In [1], it is shown that stability of the design in [8] may be lost if $d_i \not\equiv 0$. To recover stability, synchronous resets of all agents' estimates $\hat{x}_i(k)$ to their joint average are proposed. Even though resets typically occur at a lower frequency than the event triggers (6), this strategy clearly requires the exchange of the agents' estimates at resetting instants and thus extra communication.

Herein, we consider the disturbance case $d_i \not\equiv 0$ and seek to avoid all inter-agent communication besides the event-triggers (6). Firstly, to avoid communication of inputs, we use (12) in place of (11). Secondly, to avoid synchronous resetting, we formulate an LMI-design for the estimator gains L_j, which in favorable cases guarantees closed-loop stability for bounded disturbances d_i.

III. Stability Analysis

Next, the time evolution of the state trajectory and the agents' estimation errors are derived. Additionally, the inter-agent error is introduced, which allows us to express the closed-loop dynamics as a series of interconnected systems in strict feedforward form. This will allow us to derive conditions for the stability of the closed-loop system.

A. Closed-loop Dynamics

The state dynamics are obtained by combining (1) with (8), (9), and (12):

$$\begin{aligned} x(k) &= (A+BF)x(k-1) \\ &\quad - \sum_{j=1}^{N} B_j F_j e_j(k-1) + v(k-1), \end{aligned} \tag{13}$$

where the estimation error of agent j, $e_j(k)$, is defined by $e_j(k) := x(k) - \hat{x}_j(k)$.

Defining the inter-agent error to be $\epsilon_{ji}(k) := \hat{x}_j(k) - \hat{x}_i(k)$, the time evolution of agent i's estimation error is given by the combination of (1), (5), (8), and (9),

$$\begin{aligned} e_i(k) &= Ae_i(k-1) + \sum_{j=1}^{N} B_j F_j \epsilon_{ji}(k-1) \\ &- \sum_{j\in I(k)} L_j(y_j(k) - C_j\hat{x}_i(k|k-1)) - d_i(k) + v(k-1). \end{aligned}$$

Rearranging the sum over $I(k)$ yields

$$\begin{aligned} e_i(k) &= Ae_i(k-1) + \sum_{j=1}^{N} B_j F_j \epsilon_{ji}(k-1) - d_i(k) \\ &\quad + v(k-1) - \sum_{j=1}^{N} L_j(y_j(k) - C_j\hat{x}_i(k|k-1)) \\ &\quad + \sum_{j\in \bar{I}(k)} L_j(y_j(k) - C_j\hat{x}_i(k|k-1)) \end{aligned}$$

with

$$\bar{I}(k) = \{j \mid 1 \le j \le N, ||y_j(k) - C_j\hat{x}_j(k|k-1)||_2 < \delta_j\}. \tag{14}$$

Furthermore the expression $y_j(k) - C_j\hat{x}_i(k|k-1)$ can be rewritten as

$$\begin{aligned} y_j(k) - C_j\hat{x}_i(k|k-1) &= C_j(x(k) - \hat{x}_i(k|k-1)) + w_j(k) \\ &= C_j\left(Ae_i(k-1) + \sum_{m=1}^{N} B_m F_m \epsilon_{mi}(k-1) + v(k-1)\right) \\ &\quad + w_j(k) \end{aligned}$$

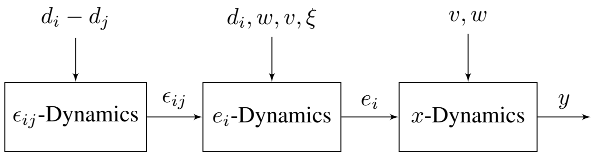

Fig. 2. Simplified block diagram representing the strict feedforward structure of the closed-loop system. The disturbances d_i, w, v, and ξ are bounded (either by assumption or by the transmission rule of the event generator).

and leads to

$$\begin{aligned} e_i(k) &= (I - LC)Ae_i(k-1) + (I - LC)v(k-1) \\ &+ (I - LC)\sum_{j=1}^{N} B_j F_j \epsilon_{ji}(k-1) + \xi(k) - d_i(k) \\ &+ \sum_{j \in \bar{I}(k)} L_j C_j (A + BF)\epsilon_{ji}(k-1) - \sum_{j=1}^{N} L_j w_j(k) \end{aligned} \tag{15}$$

with

$$\xi(k) := \sum_{j \in \bar{I}(k)} L_j (y_j(k) - C_j \hat{x}_j(k|k-1)). \tag{16}$$

Moreover, combining equations (8) and (9) implies that the inter-agent error evolves according to

$$\epsilon_{ji}(k) = (I - \sum_{m \in I(k)} L_m C_m)(A + BF)\epsilon_{ji}(k-1) + d_{ji}(k), \tag{17}$$

with $d_{ji}(k)$ defined as $d_{ji}(k) := d_j(k) - d_i(k)$. Therefore the inter-agent error is driven by a linear, switched (time varying) system subject to the bounded disturbances d_{ji}.

In summary, the closed-loop dynamics are evolving according to (13), (15), and (17). They have a strict feedforward structure as depicted in Fig. 2.

Stability is discussed using the concept of input-to-state stability (ISS), see e.g. [13], [14]. It is clear that a linear system, which is ISS is also bounded-input, bounded-output stable. Moreover, a cascaded connection of systems is ISS if each system is ISS on its own, [14][1]. Thus, the strict feedforward structure of the closed-loop system decouples the stability analysis.

B. Stability of the Inter-Agent Error

As remarked earlier, the inter-agent error is driven by a switched linear system. Switched linear systems have been an active topic of research in the past decades, see e.g. [17].

Next a conservative stability condition based on a common quadratic Lyapunov function is given. Note that it has been shown in [18] (and in [19] for continuous systems) that there exists matrices with eigenvalues of magnitude smaller than one, which do not have a common quadratic Lyapunov function, but for which the corresponding switched system subjected to arbitrary switching is still asymptotically stable. On the other hand, a switched system consisting of stable state transition matrices does not necessarily have to be stable.

For the subsequent analysis, it is convenient to define the set Π as the set of all possible permutations of $\{1, 2, \ldots, N\}$. A permutation is defined as the drawing of zero up to N elements from $\{1, 2, \ldots, N\}$ without repetition and without considering the order. Therefore, Π has cardinality $|\Pi| = 2^N$ and contains the empty set.

The next lemma is a standard result. Nonetheless a detailed proof is provided, since the derived bound on the common quadratic Lyapunov function will be used later.

Lemma 3.1: Let the matrix inequality

$$A_{\text{cl}}^{\mathrm{T}}(\Pi_i)\ PA_{\text{cl}}(\Pi_i) - P < 0 \tag{18}$$

with

$$A_{\text{cl}}(\Pi_i) := (I - \sum_{m \in \Pi_i} L_m C_m)(A + BF) \tag{19}$$

be satisfied for a positive definite matrix $P \in \mathbb{R}^{n \times n}$, $P > 0$ and for all permutations $\Pi_i \in \Pi$. Then, the inter-agent error is ISS.

Proof: Consider the positive definite quadratic Lyapunov candidate $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x) = x^{\mathrm{T}} P x. \tag{20}$$

Its time evolution along the trajectories of the inter-agent error is given by

$$\begin{aligned} V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1)) &= d_{ji}^{\mathrm{T}}(k) P d_{ji}(k) \\ &+ \epsilon_{ji}^{\mathrm{T}}(k-1)(A_{\text{cl}}^{\mathrm{T}}(I(k)) P A_{\text{cl}}(I(k)) - P)\epsilon_{ji}(k-1) \\ &+ 2 d_{ji}^{\mathrm{T}}(k) P A_{\text{cl}}(I(k))\epsilon_{ji}(k-1). \end{aligned}$$

Denoting the maximum eigenvalue of $A_{\text{cl}}^{\mathrm{T}}(I(k))PA_{\text{cl}}(I(k)) - P$ by $\lambda_{I(k)}$ allows to bound the growth $V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1))$ by

$$\begin{aligned} V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1)) &\leq \lambda_{I(k)} ||\epsilon_{ji}(k-1)||_2^2 \\ &+ 2||d_{ji}(k)||_2 ||PA_{\text{cl}}(I(k))||_2 ||\epsilon_{ji}(k-1)||_2 \\ &+ ||d_{ji}(k)||_2^2 ||P||_2. \end{aligned}$$

Completing the squares with $\alpha \in \mathbb{R}$, $\alpha > 0$, leads to

$$\begin{aligned} V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1)) &\leq (\lambda_{I(k)} + \alpha)||\epsilon_{ji}(k-1)||_2^2 \\ &+ \left(\frac{||PA_{\text{cl}}(I(k))||_2^2}{\alpha} + ||P||_2 \right) ||d_{ji}(k)||_2^2 \\ &- \left(\sqrt{\alpha}||\epsilon_{ji}(k-1)||_2 + \frac{||PA_{\text{cl}}(I(k))||_2}{\sqrt{\alpha}} ||d_{ji}(k)||_2 \right)^2. \end{aligned}$$

The evolution of V along the trajectories of the inter-agent error can therefore be bounded by

$$\begin{aligned} V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1)) &\leq \left(\lambda_{I(k)} + \alpha\right) ||\epsilon_{ji}(k-1)||_2^2 \\ &+ \left(\frac{||PA_{\text{cl}}(I(k))||_2^2}{\alpha} + ||P||_2 \right) ||d_{ji}(k)||_2^2. \end{aligned} \tag{21}$$

[1]Note that in [14] the system dynamics were assumed to be continuous, which is not the case for the inter-agent error. Nevertheless, the continuity assumption is not needed for the small gain theorem given in [14]. More about the subtleties of non-continuous system dynamics with respect to input-to-state stability and Lyapunov stability can be found in [15] or [16].

Since there exists an α such that $0 < \alpha < \min_{\Pi_i \in \Pi} |\lambda_{\Pi_i}|$, it follows that $\lambda_{I(k)} + \alpha$ can be made strictly negative. According to [16, Definition 2.2] and [16, Theorem 2.3], V is therefore a dissipative ISS-Lyapunov function, which implies input-to-state stability of the inter-agent error. ■

Remark: Throughout this article, the agents have no knowledge about the exact input u. Each agent i, $i = 1, 2, \ldots, N$, merely reconstructs u based on its current state estimate $\hat{x}_i$, i.e. $\hat{u}_i(k) = F\hat{x}_i(k)$. This is actually essential in the approach herein for stabilizing the closed-loop system and avoiding any further communication. In contrast, consider the setting where agents communicate their inputs and implement (11). In that case the evolution of the inter-agent error reduces to

$$\epsilon_{ji}(k) = (I - \sum_{m \in I(k)} L_m C_m) A \epsilon_{ji}(k-1) + d_{ji}(k).$$

A necessary condition for this system to be stable is that the matrices $(I - L_m C_m)A$ have eigenvalues of magnitude strictly less than one for each $m \in \{1, 2, \ldots, N\}$ (assuming that an arbitrary switching can occur). In order to synthesize compliant observer gains L_m, (A, C_m) must therefore be detectable for each $m \in \{1, 2, \ldots, N\}$, which would be a very strong assumption.[2] In case that each agent uses (12), the inter-agent error is governed by equation (17). Clearly the assumption of (A, C_m) being detectable is not needed, as due to the asymptotic stability of $A+BF$, $(A+BF, C_m)$ is detectable by construction.

C. Stability of the Agent Error

It will be shown next that asymptotic stability of $(I - LC)A$ together with a bounded inter-agent error is enough to guarantee input-to-state stability of the agent error.

Lemma 3.2: Let the inter-agent errors ϵ_{ji}, $j = 1, \ldots, N$, be bounded. Then, the agent error e_i is ISS if and only if the eigenvalues of $(I - LC)A$ have magnitude strictly less than one.

Proof: $\Rightarrow$: Since $(I - LC)A$ is asymptotically stable, it is enough to show that the disturbances, which the agent error is subjected to, are bounded. Hence, according to (15) it suffices to show that ξ is bounded, since d_i, v, w_j and ϵ_{ji} are bounded by assumption. By applying the triangle inequality to equation (16), it follows from (14) that

$$\begin{aligned} ||\xi(k)||_2 &\le \sum_{j \in \bar{I}(k)} ||L_j||_2 ||y_j(k) - C_j \hat{x}_j(k|k-1)||_2 \\ &\le \left(\sup_{j \in \{1,2,\ldots,N\}} ||L_j||_2 \right) \left(\sum_{j=1}^{N} \delta_j \right). \end{aligned}$$

$\Leftarrow$: Choosing disturbances parallel to an eigenvector of $(I - LC)A$ with corresponding eigenvalue having magnitude greater or equal to one shows that the agent-error is not ISS if $(I - LC)A$ has eigenvalues of magnitude greater or equal to one. ■

[2]In case (A, C_m) is detectable for each $m \in \{1, 2, \ldots, N\}$ each agent would be able to reconstruct the full state without any inter-agent communication.

D. Stability of the Closed-loop System

Using the previous results, the following condition for the closed-loop dynamics to be ISS can be established.

Theorem 3.3: Let the matrix inequalities

$$\begin{aligned} A_{\mathrm{cl}}^{\mathrm{T}}(\Pi_i) P A_{\mathrm{cl}}(\Pi_i) - P < 0 \quad \text{and} \\ ((I - LC)A)^{\mathrm{T}} Q (I - LC)A - Q < 0 \end{aligned}$$

with A_{cl} given by (19) be fulfilled for positive definite matrices $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$, $P > 0$, $Q > 0$, and for all permutations $\Pi_i \in \Pi$. Then, the closed-loop dynamics are ISS.

Proof: Note that the second matrix inequality, which is nothing but the discrete Lyapunov equation (see for example [20, p. 538]), implies asymptotic stability of $(I - LC)A$. From Lemma 3.1 and 3.2, input-to-state stability of the inter-agent error as well as the agent error follows. Since $A+BF$ is asymptotically stable by assumption, it follows from (13) that the state x is ISS. ■

IV. Synthesis of Stabilizing Observer Gains

The analysis of the previous section allowed us to come up with a criterion to check whether the closed-loop dynamics of the distributed control system are stable. Applying the Schur complement to the stability criterion can be used to generate a convex optimization problem allowing for the synthesis of stabilizing observer gains. This will be elaborated in the following.

A. LMI-Synthesis

Using the Schur complement, [21, p. 650] the LMI-conditions of Theorem 3.3 can be rewritten as[3]

$$\begin{pmatrix} P & P(I - \sum_{m \in \Pi_i} L_m C_m)(A + BF) \\ * & P \end{pmatrix} > 0$$

for all permutations $\Pi_i \in \Pi$, and

$$\begin{pmatrix} Q & Q(I - LC)A \\ A^{\mathrm{T}}(I - LC)^{\mathrm{T}} Q & Q \end{pmatrix} > 0.$$

By using $P = Q$ and introducing the change of variables $PL_m = W_m$, $m = 1, 2, \ldots, N$, the previous inequalities become linear in W_m and P. This leads to the semidefinite programming problem given by (22). Note that the vector $c \in \mathbb{R}^{|\Pi|}$, $c > 0$ is a weighting chosen by the designer, and $\lambda_{\min} \in \mathbb{R}$, $\lambda_{\min} < 0$, bounds the objective function from below (see Sec. V for a discussion).

The rationale behind the optimization is that the λ_is are tight upper bounds to the maximum eigenvalue of $A_{\mathrm{cl}}^{\mathrm{T}}(\Pi_i) P A_{\mathrm{cl}}(\Pi_i) - P$. Hence, the λ_{Π_i}s are made as small as possible, but not smaller than $c_i \lambda_{\min}$ to ensure a bounded objective function. Having all λ_{Π_i}s strictly negative implies by Theorem 3.3 that the closed-loop system is ISS.

The feasibility of the optimization problem (22) is discussed next.

Theorem 4.1: If (A, C) is detectable the optimization problem (22) is feasible.

[3]Note that $*$ refers to the transposition of the top right block, in this case $*^{\mathrm{T}} = P(I - \sum_{m \in \Pi_i} L_m C_m)(A + BF)$.

$$\begin{aligned}
&\underset{\lambda}{\text{minimize}} \quad \sum_{i=1}^{|\Pi|} c_i \lambda_{\Pi_i} \quad \text{s.t.} \\
&P \in \mathbb{R}^{n\times n},\ P = P^{\mathrm{T}},\ \lambda = (\lambda_{\Pi_1}, \lambda_{\Pi_2}, \dots, \lambda_{\Pi_{|\Pi|}}) \in \mathbb{R}^{|\Pi|}, \\
&W_i \in \mathbb{R}^{n\times p_i},\ i = 1, 2, \dots, N, \\
&\lambda_{\Pi_i} \geq c_i \lambda_{\min},\ i = 1, 2, \dots, |\Pi|, \\
&\begin{pmatrix} P & (P - \sum_{m\in\Pi_i} W_m C_m)(A + BF) \\ * & P + \lambda_{\Pi_i} I \end{pmatrix} \geq 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall\ \Pi_i \in \Pi, \\
&\begin{pmatrix} P & (P - \sum_{m=1}^{N} W_m C_m) A \\ * & P \end{pmatrix} > 0.
\end{aligned} \tag{22}$$

Proof: Since (A, C) is detectable there exists a feedback matrix L such that $(I - LC)A$ is asymptotically stable. This implies the existence of a matrix $P \in \mathbb{R}^{n\times n}$ with $P = P^{\mathrm{T}} > 0$, which fulfills

$$\begin{pmatrix} P & P(I - \sum_{m=1}^{N} L_m C_m) A \\ * & P \end{pmatrix} > 0.$$

By substituting $W_m = PL_m$, $m = 1, 2, \dots, N$, it follows that the last matrix inequality of (22) is fulfilled. Taking the Schur complement of the remaining matrix inequalities yields $A_{\mathrm{cl}}^{\mathrm{T}}(\Pi_i) P A_{\mathrm{cl}}(\Pi_i) - P - \lambda_{\Pi_i} I < 0$. Clearly, the λ_{Π_i} ($\lambda_{\Pi_i} \geq c_i \lambda_{\min}$) can be chosen larger than $||A_{\mathrm{cl}}^{\mathrm{T}}(\Pi_i) P A_{\mathrm{cl}}(\Pi_i) - P||_2$ such that the remaining inequalities are satisfied for all $\Pi_i \in \Pi$. ■

In general, there is no guarantee that negative λ_{Π_i} can be found, as this would imply the existence of a common quadratic Lyapunov function for the inter-agent error. Establishing statements about the existence of a common quadratic Lyapunov function for a switched linear system has been proven to be difficult, see e.g. [17].

Instead, a different approach is pursued: Without imposing $\lambda < 0$ the optimization problem (22) is guaranteed to be feasible according to Theorem 4.1. Two cases can be distinguished:

1) If all λ_{Π_i} are negative the closed-loop system is guaranteed to be ISS.
2) If at least one λ_{Π_i} is greater or equal to 0, additional communication is needed. In the next section, IV-B, a strategy is presented to bound the inter-agent error and thus ensure input-to-state stability of the closed-loop system. This can be seen as generalization of the periodic reset strategy suggested in [1].

B. Boundedness of the Inter-Agent Error in Case $\lambda_{\Pi_i} \geq 0$

This paragraph discusses the case where at least one λ_{Π_i} is greater or equal zero. It will be shown that an upper bound to the quadratic Lyapunov candidate (20) allows for deriving a periodic reset strategy as suggested in [1], which guarantees input-to-state stability of the closed-loop system.

From (21), it can be deduced that

$$\begin{aligned}
V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1)) &\leq (\lambda_{I(k)} + \alpha)||\epsilon_{ji}(k-1)||_2^2 \\
&+ \left(\frac{||PA_{\mathrm{cl}}(I(k))||_2^2}{\alpha} + ||P||_2\right) D^2
\end{aligned}$$

with the constant $\alpha \in \mathbb{R}$, $\alpha > 0$, and D an upper bound to the disturbances $d_{ji}(k)$, i.e. $||d_{ji}(k)||_2 < D$ for all k. The bound can be further relaxed by using $\lambda_{\max} := \max_{\Pi_i\in\Pi} \lambda_{\Pi_i}$ and $\gamma_{\max} := \max_{\Pi_i\in\Pi} ||PA_{\mathrm{cl}}(\Pi_i)||_2$. Combined with the fact that $V(\epsilon_{ji}(k-1)) \geq \sigma_{\min}(P)||\epsilon_{ji}(k-1)||_2^2$ this yields

$$\begin{aligned}
V(\epsilon_{ji}(k)) - V(\epsilon_{ji}(k-1)) &\leq \frac{\lambda_{\max} + \alpha}{\sigma_{\min}(P)} V(\epsilon_{ji}(k-1)) \\
&+ \left(\frac{\gamma_{\max}^2}{\alpha} + ||P||_2\right) D^2,
\end{aligned}$$

where $\sigma_{\min}(P)$ denotes the minimum singular value of P. Since there is a $\lambda_{\Pi_i} \geq 0$ and $\alpha > 0$ the ratio $\frac{\lambda_{\max}+\alpha}{\sigma_{\min}(P)}$ is positive. For tightening the upper bound, the right hand side can be minimized with respect to α, which results in

$$\alpha_{\min}(V) = \sqrt{\frac{\sigma_{\min}(P)}{V}}\gamma_{\max}^2 D > 0. \tag{23}$$

Hence an estimate $\hat{V}(k)$, with $\hat{V}(k) \geq V(\epsilon_{ji}(k))$ can be determined by

$$\begin{aligned}
\hat{V}(k) &= \left(1 + \frac{\lambda_{\max} + \alpha_{\min}(\hat{V}(k-1))}{\sigma_{\min}(P)}\right) \hat{V}(k-1) \\
&+ \left(\frac{\gamma_{\max}^2}{\alpha_{\min}(\hat{V}(k-1))} + ||P||_2\right) D^2,
\end{aligned} \tag{24}$$

for $k = 1, 2, \dots$ and $\hat{V}(0) = 0$ (assuming that the agents are initialized with the same state estimate). Note that the matrix P is obtained by solving (22) and allows to precalculate $\sigma_{\min}(P)$, $\lambda_{\max}$ and $\gamma_{\max}$.

As soon as $\hat{V}$ exceeds the predefined threshold $V_{\max}$, i.e. $\hat{V}(k) > V_{\max}$, a communication is triggered and the different agents' state estimates are set to a common value to reset the inter-agent error implying $\epsilon_{ji}(k) = 0$ and $\hat{V}(k) = 0$. Clearly, this reset strategy bounds the inter-agent error since $V_{\max} \geq V(k) \geq \sigma_{\min}(P)||\epsilon_{ji}(k)||_2^2$ for all k. By the strict feedforward structure of the closed-loop dynamics, this implies input-to-state stability of the state x and the agents' estimator errors e_i, $i = 1, 2, \dots, N$. Note that there are many potential different reset strategies such as a majority vote, the mean, etc. Since the evolution of $\hat{V}(k)$ is time-independent the time instants k_{reset_i} where $\hat{V}(k_{\mathrm{reset}_i})$ exceeds $V_{\max}$, $i = 1, 2, \dots$ can be precalculated. This amounts in a periodic reset and extends the procedure presented in [1] by providing a strategy for chosing the reset period.

Note that the estimate $\hat{V}$ grows faster with a larger $\lambda_{\max}$. Hence, minimizing the λ_{Π_i} by solving (22) still reduces the growth of $\hat{V}$ and therefore the communication, although $\lambda_{\Pi_i} < 0$ is not obtained for each $\Pi_i \in \Pi$.

V. SIMULATION EXAMPLE

In this section we present a simulation example to illustrate the distributed estimation and control framework introduced in the previous sections. In particular, we introduce random packet drops and different initial conditions causing the agents' state estimates to differ. To evaluate and compare the performance, the simulation example is according to [1].

A. Simulation Model

In the following, the inverted pendulum system depicted in Fig. 3 is considered, where φ_1 and φ_2 parametrize the inclination of the first and second arm, and θ the inclination of the inverted pendulum. Taking the same physical parameters as in [1] leads to the following linearized system dynamics:

$$x(k+1) = \begin{pmatrix} 1.0007 & 0.0100 & -0.0001 & -0.0005 & -0.0001 & -0.0012 \\ 0.1492 & 1.0007 & -0.0151 & -0.0462 & -0.0151 & -0.1231 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0.0005 & 0.0012 \\ 0.0461 & 0.1230 \\ 0.0100 & 0 \\ 1 & 0 \\ 0 & 0.0100 \\ 0 & 1 \end{pmatrix} u(k),$$

where $x^{\mathrm{T}} = (\theta, \dot{\theta}, \varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)$. The desired angular velocities of the pendulum arms, i.e. $\dot{\varphi}_{1_{\text{des}}}$ and $\dot{\varphi}_{2_{\text{des}}}$ are regarded as inputs. This can be realized by a substantially faster inner control loop, which tracks the desired angular rates such that $\dot{\varphi}_i(k+1) = \dot{\varphi}_{i_{\text{des}}}(k)$, $i = 1, 2$, see [22] for details.

The lower control unit is called *agent 1* and the upper *agent 2*. Agent 1 computes the desired angular velocity $u_1 = \dot{\varphi}_{1_{\text{des}}}$ and has access to noisy measurements $\varphi_1 + n_{\varphi_1}$, $\dot{\varphi}_1 + n_{\dot{\varphi}_1}$, and $\dot{\theta} + n_{\dot{\theta}}$. Agent 2 computes the desired angular velocity $u_2 = \dot{\varphi}_{2_{\text{des}}}$ and measures $\varphi_2 + n_{\varphi_2}$ and $\dot{\varphi}_2 + n_{\dot{\varphi}_2}$. The disturbances n_{φ_i}, $n_{\dot{\varphi}_i}$, and $n_{\dot{\theta}}$ are independent, zero mean and uniformly distributed with variances $\sigma^2_{\varphi_i} = (0.05\,°)^2$, $\sigma^2_{\dot{\varphi}_i} = (0.1\,°/\mathrm{s})^2$, and $\sigma^2_{\dot{\theta}} = (0.24\,°/\mathrm{s})^2$, $i = 1, 2$.

Note that the system is neither observable nor controllable for each agent on its own. Nevertheless, it is controllable and observable for both agents together.

The controller F is found via an LQ regulator approach and is given by

$$F = \begin{pmatrix} 212.5872 & 55.0168 & -19.3450 & -2.5374 & -23.5728 & -6.7664 \\ -84.9883 & -22.0881 & 6.4894 & 1.0187 & 6.3579 & 2.7166 \end{pmatrix}.$$

B. Design of Observer Gains

The vector $c = (2, 1, 4, 0.5)^{\mathrm{T}}$ is used for weighting the different eigenvalues $\lambda_{\Pi_1}, \lambda_{\Pi_2}, \ldots, \lambda_{\Pi_4}$ defined in (22). The eigenvalue λ_{Π_1} describes the case where only agent 1 communicates its measurements ($\Pi_1 = \{1\}$), λ_{Π_2} the case where only agent 2 communicates its measurements ($\Pi_2 = \{2\}$), λ_{Π_3} the case where both agents communicate their measurements ($\Pi_3 = \{1, 2\}$), and λ_{Π_4} the communication free case ($\Pi_4 = \{\}$). Since the open-loop system is unstable, the case where no communication occurs is expected to happen rarely and therefore the weighting c_4 is chosen to be the lowest. In case that both agents communicate, a drastic reduction in the inter-agent error is desirable, which leads to the comparably large value of $c_3 = 4$. Additionally, the case where agent 1 communicates its measurements is expected to happen more frequently (agent 1 measures $\dot{\theta}$) and therefore c_1 is chosen to be higher than c_2. The constant

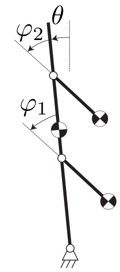

Fig. 3. Inverted pendulum model: The pendulum is stabilized by the relative motion of its two arms.

$\lambda_{\min}$ bounding the objective function from below is chosen as $\lambda_{\min} = -1$. By solving the problem (22) the observer gains L_1 and L_2 for agent 1 and 2 are obtained. The eigenvalues λ_{Π_i} are found to be $\lambda_{\Pi_1} = -2$, $\lambda_{\Pi_2} = -1$, $\lambda_{\Pi_3} = -4$ and $\lambda_{\Pi_4} = -0.5$, which indicates an active lower bound constraint $\lambda_{\Pi_i} \geq \lambda_{\min} c_i$, $i = 1, 2, \ldots, 4$. Since all λ_{Π_i} are negative the additional reset strategy described in Sec. IV-B is not needed for guaranteeing input-to-state stability of the closed-loop system.

C. Simulation Results

The communication thresholds δ_1 and δ_2 are set to $\delta_1 = 8 \cdot 10^{-3}$ and $\delta_2 = 5 \cdot 10^{-3}$. Each agent knows its initial module inclination, but not the initial inclination of the pendulum. Thus, the initial conditions are chosen to be $x(0) = (1\,°, 0, 0.1\,°, 0, -0.1\,°, 0)^{\mathrm{T}}$, $\hat{x}_1(0) = (0, 0, 0.1\,°, 0, 0, 0)^{\mathrm{T}}$ and $\hat{x}_2(0) = (0, 0, 0, 0, -0.1\,°, 0)^{\mathrm{T}}$. Moreover, a packet drop[4] is assumed to occur with a probability of 2%, i.e. on average one in 50 measurements is lost. The system is therefore subjected to disturbances coming from the non-zero initial conditions, the non-zero initial inter-agent error, the measurement noise, and the random packet drops. The resulting closed-loop trajectory of the pendulum inclination angle is depicted in Fig. 4 together with the agents' estimation errors and the estimated communication per agent. The communication rates $\mathcal{R}_1$ and $\mathcal{R}_2$ are normalized such that 1 corresponds to an agent communicating at every time instant.

It can be observed that the system reaches steady state after around 7 s. At steady state a total communication $\mathcal{R} = \frac{1}{2}(\mathcal{R}_1 + \mathcal{R}_2)$ of approximately 19% is observed, together with a root-mean-squared inclination angle error of $0.21\,°$.

Using the approach proposed by [1] the steady state inclination error can be reduced to $0.06\,°$ and the total communication to around 15% (without taking the permanent communication of the input into account), depending on the tuning of the Luenberger observer gains. There are two reasons explaining the performance improvement: 1) In [1] the exact input u is assumed to be known by both agents; and 2) the synthesis problem (22) does not include any performance measure since it accounts only for stability. However, the observer gains obtained from the centralized

[4]Even though the disturbance d_i in (9) is not guaranteed to be bounded when representing packet drops (see [1] for details), the following simulation results demonstrate the effectiveness of the method also in this case.

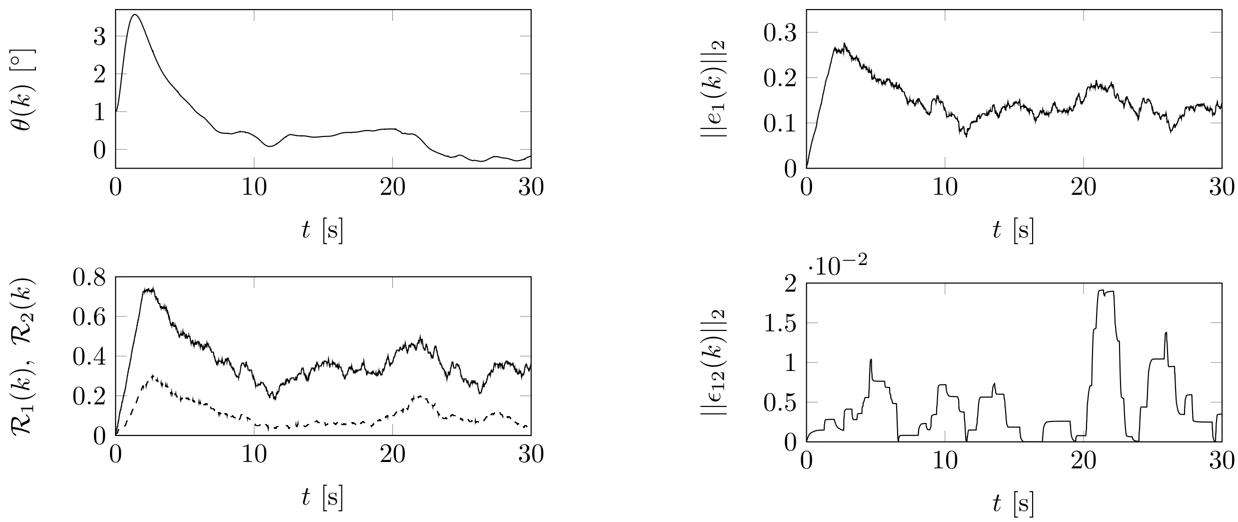

Fig. 4. Top left: time history of pendulum inclination angle $\theta(k)$ using a sampling time of 1 ms. Top right: 2-norm of agent 1's estimation error (after smoothing with a 200-sample moving average filter). Bottom left: estimation of the communication $\mathcal{R}_1$ of agent 1 (solid) and $\mathcal{R}_2$ of agent 2 (dashed) via a 200-sample moving average filter. Bottom right: 2-norm of the inter-agent estimation error (after smoothing with a 200-sample moving average filter).

synthesis approach according to [1] lead to an unstable inter-agent error, which necessitates the use of a reset strategy.

VI. Conclusion

This paper presented a new approach to the synthesis of stabilizing observer gains for a linear event-based control system. In contrast to [1], the resulting control and estimation algorithm is not required to have exact knowledge of the input u. Additionally, it has been shown that under favorable circumstances (negative λ_{Π_i}) the periodic reset of the agents' estimates as proposed in [1] is not needed.

The framework can be extended to treat the case where every agent continuously updates its state estimate with its own measurements. This does not lead to additional communication, and it can be shown to preserve closed-loop stability. The observer gain synthesis is formulated as a semidefinite program and therefore it can be easily combined with a $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance measure to improve the closed-loop behavior. Also, relaxations of the design are possible to reduce the computational complexity for a large number of agents. A rigorous discussion of these aspects is future work.

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